Algebraic analysis of McCarthy logic

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1 General presentation of the project and state of the art

The idea of using many-valued logics to reason about partial predicates in algorithms arose quite early in computer science. After more than sixty years, the systems built out of Kleene's "strong" and "weak" connectives [10] and McCarthy's three-valued logic [13] still remain the two main proposals to model partiality in a many-valued setting.
Over the time, many three- and four-valued calculi, the most extending or combining the logics of Kleene and McCarthy[1], have been developed and found application in a broad range of areas. Here are some significant examples.

Area	Context	Applications
Software engineering	Software specification and validation	General purpose formal specification languages such as VDM-SL[2], RSL[3], Z[4], and OCL[5] [9].
Cybersecurity	Access control policy specification	Semantical analysis of XACML v3.0[6] [7] and its rivals (e.g., PTaCL [6]).
Concurrency theory	Accounts of computation errors in concurrency	Combination of the logic BM_3 and ACP [2]

[1] A remarkable example is the MK-calculus [3], developed in the framework of the MetaSoft Project.
[2] https://www.iso.org/standard/22988.html.
[3] https://ntrs.nasa.gov/citations/19930003157.
[4] https://www.iso.org/standard/21573.html.
[5] https://www-st.inf.tu-dresden.de/ocl/.
[6] https://docs.oasis-open.org/xacml/3.0/xacml-3.0-core-spec-os-en.html.

Area	Context	Applications
Database theory	Representation of incomplete information in DBMSs	Many-valued approach to the problems brought by NULL values in relational DBs [8, 5].

McCarthy logic stems from the following tables extending classical negation, conjunction, and disjunction.

$\neg$	
1	0
u	u
0	1

$\wedge$	1	u	0
1	1	u	0
u	u	u	u
0	0	0	0

$\vee$	1	u	0
1	1	1	1
u	u	u	u
0	1	u	0

The McCarthy tables

Since this calculus was specifically designed to describe sequential computation, $\wedge$ and $\vee$ are non-commutative. Complex formulas are thus to be evaluated left-to-right, in a "lazy" way. These features made McCarthy logic maybe unattractive compared to Kleene systems, which have been thoroughly studied via algebraic methods. Instead, very little progress has been made in algebraic investigations of McCarthy logic. The goal of this project is to fill this gap.

2 RESEARCH OBJECTIVES AND METHODOLOGY

To the best of my knowledge, *MC-algebras* are the only candidate so far proposed to be the algebraic counterpart of McCarthy logic. Such structures were introduced and axiomatized by B. Konikowska in [11], where representation theorems for two classes of MC-algebras (atomically selective and countable selective algebras) are provided. The purpose of my research is an in-depth analysis of such structures within the framework of universal algebra [1]. First observe that MC-algebras admit an equational axiomatization, hence they form a variety, which we call $\mathfrak{MC}$. From a purely algebraic standpoint, Konikowska's results are not fully satisfactory in light of the following open problems, which I would like to address:

i. Konikowska's axioms are sound w.r.t. McCarthy tables, but completeness still needs to be (dis)proven;

ii. the inner structure of MC-algebras has not yet been made explicit (e.g., despite the asymmetry of $\wedge$ and $\vee$, MC-algebras do not appear to be skew Boolean algebras [12] nor do they appear to have a skew lattice reduct);

iii. structure theorems for $\mathbf{L}_v(\mathfrak{MC})$, the lattice of subvarieties of $\mathfrak{MC}$ are absent in the literature. Moreover, a closer look at the axioms for MC-algebras reveals some connections between $\mathfrak{MC}$ and the variety $\mathfrak{IBSL}$ of *involutive bisemilattices*. $\mathfrak{IBSL}$ plays an important role in the algebraic semantics of the logic PWK [4]. Therefore, a study of the interrelation of $\mathfrak{MC}$ and $\mathfrak{IBSL}$ may be useful for the development of new combinations of Kleene- and McCarthy-style systems.

3 EXPECTED RESULTS

The outcomes I expect from my research are the following:

1. axiomatic characterization of MC-algebras in different similarity types and identification of at least one axiom system sound and complete w.r.t. McCarthy tables;
2. development of a comprehensive representation theory for MC-algebras;
3. application of the obtained results in the four areas listed in section 1.

4 REFERENCES

[1] BERGMAN, C. *Universal Algebra. Fundamentals and Selected Topics*, 2011.

[2] BERGSTRA, J., PONSE, A. Bochvar-McCarthy Logic and Process Algebra. *Notre Dame J. Form. Log.* (Fall 1998). DOI: 10.1305/ndjfl/1039118863.

[3] BLIKLE, A. *MetaSoft Primer. Towards a Metalanguage for Applied Denotational Semantics* (*LNCS* 288), 1987. DOI: 10.1007/3-540-18657-3.

[4] BONZIO, S. et al. *Logics of Variable Inclusion*, 2022. DOI: 10.1007/978-3-031-04297-3.

[5] CONSOLE M. et. al. Propositional and predicate logics of incomplete information. *Artif. Intell.* (Jan. 2022). DOI: 10.1016/j.artint.2021.103603.

[6] Crampton, J., Morisset, C. PTaCL: A Language for Attribute-Based Access Control in Open Systems. In *Principles of Security and Trust* (*LNCS* 7215), 2012. DOI: 10.1007/978-3-642-28641-4_21.

[7] Dewi Puspa Kencana Ramli, C. et al. The logic of XACML. *Sci. Comput. Program.* (Apr. 2014). DOI: 10.1016/j.scico.2013.05.003.

[8] Grahne, G., and Moallemi, A. A useful four-valued database logic. In *Proceedings of the 22nd International Database Engineering & Applications Symposium* (IDEAS 2018), 2018. DOI: 10.1145/3216122.3216157.

[9] Hähnle, R. Many-Valued Logic, Partiality, and Abstraction in Formal Specification Languages. *Logic J. IGPL* (Jul. 2005). DOI: 10.1093/jigpal/jzi032.

[10] Kleene, S.C. *Introduction to Metamathematics*, 1952.

[11] Konikowska, B. McCarthy algebras: a model of McCarthy's logical calculus. *Fundam. Inform.* (1996). DOI: 10.3233/FI-1996-26205.

[12] Leech, J. Skew Boolean Algebras. *Algebra Univers.* (Dec. 1990). DOI: 10.1007/BF01188995.

[13] McCarthy, J. A basis for a mathematical theory of computation. In *Computer Programming and Formal Systems*, 1963. DOI: 10.1016/S0049-237X(08)72018-4.

5 Description of the research in the three-year period

- First year - Study of the existing literature on McCarthy logic and its semantics. First contributions to the theory of MC-algebras. Attendance of courses, lessons, summer/winter schools to complete my theoretical computer science skills.
- Second year - Article writing and submission to journal, workshops and conferences. Preliminary organization of the thesis work.
- Third year - Research work, thesis writing.

6 Keywords and research areas

Keywords: McCarthy logic, MC-algebras, sequential computation, computation errors, formal specification, process algebra, incomplete information
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ERC fields: PE1_1, PE1_2, PE6_4, SH4_13
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